On Hilbert Transform and its application to assessment of electrical signal waveform distortion

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Abstract - **In order to achieve full description of the distorted signals applied method of analysis should take into consideration two kind of desirable parameters: those which characterizes time behavior of the waveform, and those which gives information about frequency components. This paper follows by the idea of Hilbert Transform (HT) which allows to construct analytic signal and introduce the term of instantaneous amplitude (IA), instantaneous phase (IP) and instantaneous frequency (IF). Unfortunately, application of HT, especially in point of instantaneous frequency, gives meaningful results for restrictive class of signals. The accurate representation is achieved for monocomponents signals. Practical application of HT can be preprocessed by selected signal decomposition method. This paper provides some analytic derivation of HT and its digital application for not decomposed and decomposed signals.**

I. Introduction

Dynamic complex phenomena require extended performance of applied signal processing methods. One-dimensional Fourier spectrum analysis, however very useful and fast digital applicable, can be insufficient, providing only general information about extracted signal components, with loss of its time-varying nature. The frequency analysis can provide the frequency details, but unfortunately, we don't know when the frequency changes occurred. In case of dynamic phenomena the assumption of the stationarity can not be fulfilled. Signal processing has provided the solution for given problem using two techniques: joint time-frequency analysis (JTFA) or decomposition of the signal and calculation of instantaneous amplitude and instantaneous frequency for particular signal components. This paper is concentrated on mechanism and meaning of Hilbert transform (HT) for calculation of instantaneous magnitude and instantaneous frequency.

Hilbert transform allows constructing orthogonal signal to the given one. Having the pair of signal and its orthogonal form we can define complex analytic signal with real part as given signal and imaginary part as the orthogonal effect of Hilbert transformation. Obtained analytic signal is complex function of time. For every time instant we can recalculate amplitude as well as phase of the local, in point of time, complex value. It leads to new characteristics, which represents changes of amplitude and changes of phase in time. Derivative of phase in time is defined as instantaneous frequency [1],[2],[3],[4].

Successful application of Hilbert transform is possible only for narrow range of signals. The meaningful results are achieved in case of monocomponents signals. Thus, in practical cases, the HT is applied not strictly for investigated signal, usually containing many components, but for single selected component, obtained using some signal decomposition method. There are several signal decomposition methods including orthogonal polynomial expansion, Fourier expansion or wavelet expansion. Last work of Huang brought new idea of signal expansion into the set of so called intrinsic mode function (IMF), which represents the oscillatory mode embedded in the signal. The algorithm, which serves the idea of representation of the signal by the set of IMF oscillatory modes, is called empirical mode decomposition (EMD) and was derived by Huang et al. Obtained elements of expansion can be putted through an examination by Hilbert transform in order to track instantaneous amplitude and instantaneous frequency of particular components of the expansion. Described approach is known as Hilbert Huang Transform (HHT) [5],[6],[7]. Some selected application of Hilbert spectrum has found its place also in electrical engineering [8],[9],[10],[11].

The aim of presented studies is to describe Hilbert transform technique and to test its usefulness for fundamental benchmark cases of monocomponent and pseudomonocomponent signals. In order to investigate the methods several experiments were performed supported by analytic derivation and digital simulation. One of the contributions of this paper is the exploration of Hilbert transform in comparison with its continuous and digital realization. The intention of presented results is to familiarize with Hilbert engine as crucial element in process of calculation of instantaneous amplitude, phase and frequency.

II. Hilbert Transform and Instantaneous parameters

The Hilbert transform of given real signal $x(t)$ allows to obtain its orthogonal form $y(t)$ as follows [1],[2],[3],[4]:

$$\begin{aligned} y(t) &= H\{x(t)\} = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{x(\tau)}{t-\tau}d\tau \\ x(t) &= H^{-1}\{y(t)\} = -\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{y(\tau)}{t-\tau}d\tau \end{aligned} \qquad (1)$$

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Above definition can be presented in convolution form:

$$
\begin{aligned}
y(t) &= \frac{1}{\pi t} * x(t) \\
x(t) &= \left(-\frac{1}{\pi t}\right) * y(t)
\end{aligned}
\tag{2}
$$

Having a pair of orthogonal components we can defined complex analytic signal $z(t)$ as:

$$
\begin{aligned}
& z(t) = x(t) + jy(t) = |z(t)| \cdot e^{j\psi(t)} \\
& |z(t)| = \sqrt{(x(t))^2 + (y(t))^2}; \psi(t) = arg(z(t))
\end{aligned}
\tag{3}
$$

Here reveals definition of instantaneous amplitude (IA) recognized as:

$$
IA(t) = |z(t)| = \sqrt{(x(t))^2 + (y(t))^2} \tag{4}
$$

Simultaneously, we can distinguish instantaneous phase (IP) as phase of analytic signal or as imaginary part of natural logarithm of complex analytic function:

$$
\begin{aligned}
& IP(t) = \psi(t) = arg(z(t)) \text{ or } IP(t) = Im\{ln\, z(t)\} \\
& ln(z(t)) = ln\left(\left|z(t) e^{j\psi(t)}\right|\right) = ln(|z(t)|) + j\psi(t)
\end{aligned}
\tag{5}
$$

Finally, instantaneous angular frequency can be defined as derivative of instantaneous phase in time.

$$
\begin{aligned}
& \omega(t) = \frac{d\psi(t)}{dt} \\
& or\ \omega(t) = Im\left\{(ln(z(t)))'\right\} = Im\left\{\frac{z'(t)}{z(t)}\right\} \\
& \omega(t) = \frac{x(t)y'(t) - y(t)x'(t)}{x^2(t) + y^2(t)}
\end{aligned}
\tag{6}
$$

Additional scaling by factor $1/2\pi$ leads to instantaneous frequency (IF):

$$
IF(t) = f(t) = \frac{1}{2\pi}\omega(t) = \frac{1}{2\pi}\frac{d\psi(t)}{dt} \tag{7}
$$

Application of Hilbert transform in definition form (1) is difficult. Effective way utilizes its convolution form (2) and some relations between Fourier transform and the convolution. Required in this approach is the Fourier transform of investigated signal $x(t)$. Having:

$$
\begin{aligned}
& F\{x(t)\} = X(j\omega) \\
& F\left\{\frac{1}{\pi t}\right\} = -j\, sgn\, \omega = \begin{cases} -j, for\ \omega > 0 \\ 0, for\ \omega = 0 \\ j, for\ \omega < 0 \end{cases}
\end{aligned}
\tag{8}
$$

we can recalculate Fourier transform of orthogonal component $y(t)$ as:

$$
\begin{aligned}
& F\{y(t)\} = Y(j\omega) = F\left\{\frac{1}{\pi t}\right\} \cdot F\{x(t)\} \\
& Y(j\omega) = -j\, sgn\, \omega \cdot X(j\omega)
\end{aligned}
\tag{9}
$$

Hilbert transform of given signal can be now uncovered as inverse Fourier transform:

$$
y(t) = F^{-1}\{Y(j\omega)\} \tag{10}
$$

Construction of complex analytic signal z(t) according to (3) is the initial point for further calculation of instantaneous amplitude and frequency. Selected analytic derivation and computer application of instantaneous amplitude and instantaneous frequency calculation was performed.

A. Monocomponent sinusoidal case

The area of interests includes sinusoidal signals, mainly studied in electrical engineering fundamentals. Thus, firstly we have considered monocomponent sinusoidal signal $x(t) = A_1 sin(\omega_1 t + \psi_1)$, supported by analytic derivation presented in Table I. Fig 1 presents the numerical results when the investigated signal is 50Hz sinusoidal function with amplitude equals one and phase initial equals 0. Referring to the comments in Table I limitation of phase from $-\pi$ to π makes instantaneous phase a linear periodical function instead of linear continuous function. Further numerical calculation of instantaneous frequency according to (7) brings local delta Dirac components. Its localization depends on the placement of "artificial" phase jump from $-\pi$ to π, which can be associated with phase initial of investigated sine signal. Mentioned influence depicts Fig. 2.

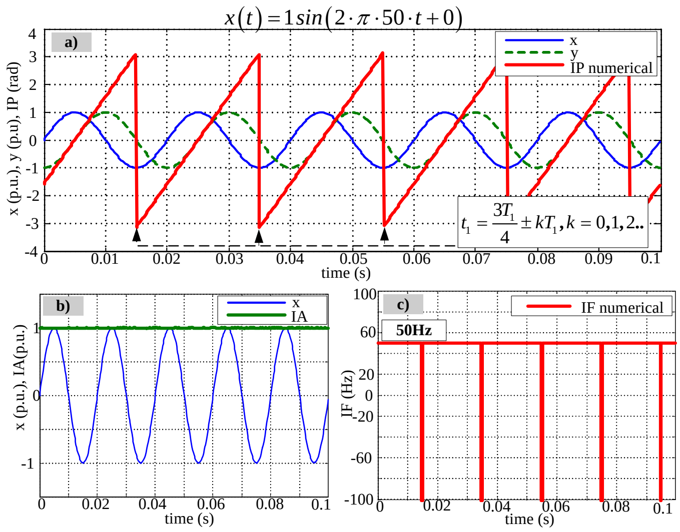

Fig. 1. Example of numerical calculation of Hilbert transform: (a) investigated signal $x(t)$ and its orthogonal form $y(t)$ with instantaneous phase IP(t);(b) signal and instantaneous amplitude IA(t);(c) instantaneous frequency IF(t)

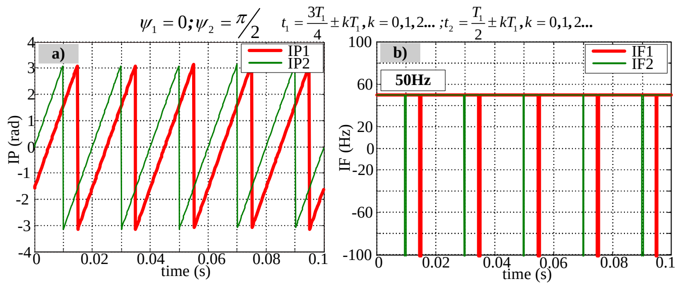

Fig. 2. Influence of phase initial of sinusoidal component on numerical application of Hilbert transform: (a) instantaneous phase IP(t); (c) instantaneous frequency IF(t)

TABLE I

ANALYTIC DERIVATION OF INSTANTANEOUS AMPLITUDE AND FREQUENCY OF: $x(t) = A_1 \sin(\omega_1 t + \psi_1)$

Step 1: Derivation of Hilbert transform of orthogonal component *y*(t) using Fourier form of given signal *x*(*t*) and formulas (9) and (10):

$$x(t) = A_1 \boldsymbol{sin}(\omega_1 t + \psi_1) \xrightarrow{Fourier} X(j\omega) = jA_1\pi\left[e^{-j\psi_1}\delta(\omega+\omega_1) - e^{j\psi_1}\delta(\omega-\omega_1)\right]$$

$$Y(j\omega) = -j\,\boldsymbol{sgn}\,\omega \cdot X(j\omega) = -j\,\boldsymbol{sgn}\,\omega \cdot jA_1\pi\left[e^{-j\psi_1}\delta(\omega+\omega_1) - e^{j\psi_1}\delta(\omega-\omega_1)\right] = \boldsymbol{sgn}\,\omega A_1\pi\left[e^{-j\psi_1}\delta(\omega+\omega_1) - e^{j\psi_1}\delta(\omega-\omega_1)\right]$$

$$= -A_1\pi\left[e^{-j\psi_1}\delta(\omega+\omega_1) + e^{j\psi_1}\delta(\omega-\omega_1)\right] \xrightarrow{Inverse\ Fourier} y(t) = F^{-1}\{Y(j\omega)\} = -A_1 \boldsymbol{cos}(\omega_1 t + \psi_1)$$

Step 2: Definition of instantaneous amplitude and phase referring to analytic signal *z*(*t*):

$$IA(t) = |z(t)| = \sqrt{\left(A_1 \boldsymbol{sin}(\omega_1 t + \psi_1)\right)^2 + \left(-A_1 \boldsymbol{cos}(\omega_1 t + \psi_1)\right)^2} = A_1$$

Comments: Instantaneous amplitude of monocomponent signal confirms constant character of its amplitude.

$$IP(t) = \psi(t) = \boldsymbol{arg}(z(t)) = arctg\left(\frac{-A_1 \boldsymbol{cos}(\omega_1 t + \psi_1)}{A_1 \boldsymbol{sin}(\omega_1 t + \psi_1)}\right) = arctg\left(-\boldsymbol{ctg}(\omega_1 t + \psi_1)\right) = arctg\left(tg\left(\omega_1 t + \psi_1 - \frac{\pi}{2}\right)\right) = \omega_1 t + \psi_1 - \frac{\pi}{2}$$

Comments: Instantaneous phase is a linear function of time with coefficient equals ω_1. In practical or digital realization the angle is limited to $\langle -\pi, \pi \rangle$. Introducing mentioned limitation given linear function becomes periodical linear function with period $T_1 = 2\pi / \omega_1$. We can distinguish the time constant when the function of instantaneous phase obtains the border value π and then jumps to $-\pi$:

$$\omega_1 t + \psi_1 - \frac{\pi}{2} = \frac{2\pi}{T_1} t_1 + \psi_1 - \frac{\pi}{2} = \pi \rightarrow t_1 = \left(\pi + \frac{\pi}{2} - \psi_1\right)\frac{T_1}{2\pi} = \left(\frac{3\pi}{2} - \psi_1\right)\frac{T_1}{2\pi} \pm kT_1, k = 0,1,2..$$

Step 3: Calculation of instantaneous angular frequency:

Method 1 using derivative and real and imaginary part of analytic signal:

$$\omega(t) = \frac{d\psi(t)}{dt} = \frac{x(t)y'(t) - y(t)x'(t)}{x^2(t) + y^2(t)} == \frac{A_1 \boldsymbol{sin}(\omega_1 t + \psi_1) A_1 \boldsymbol{sin}(\omega_1 t + \psi_1)\omega_1 + A_1 \boldsymbol{cos}(\omega_1 t + \psi_1)\omega_1 A_1 \boldsymbol{cos}(\omega_1 t + \psi_1)}{A_1^2 \boldsymbol{sin}^2(\omega_1 t + \psi_1) + A_1^2 \boldsymbol{cos}^2(\omega_1 t + \psi_1)} = \frac{A_1^2\omega_1}{A_1^2} = \omega_1$$

Method 2 using combined derivative and trigonometry approach:

$$\boldsymbol{tg}\,\psi(t) = \frac{y(t)}{x(t)} = \frac{-\boldsymbol{cos}(\omega_1 t + \psi_1)}{\boldsymbol{sin}(\omega_1 t + \psi_1)} = -\boldsymbol{ctg}(\omega_1 t + \psi_1) \xrightarrow{d/dt} \frac{1}{\boldsymbol{cos}^2\psi(t)}\psi'(t) = -\left(\frac{-1}{\boldsymbol{sin}^2(\omega_1 t + \psi_1)}\right)\omega_1 \rightarrow \psi'(t) = \frac{\boldsymbol{cos}^2\psi(t)}{\boldsymbol{sin}^2(\omega_1 t + \psi_1)}\omega_1$$

$$\boldsymbol{cos}^2\psi(t) = \frac{1}{1 + \boldsymbol{tg}^2\psi(t)} = \frac{1}{1 + \boldsymbol{ctg}^2(\omega_1 t + \psi_1)}\boldsymbol{;}\ \boldsymbol{sin}^2(\omega_1 t + \psi_1) = \frac{1}{1 + \boldsymbol{ctg}^2(\omega_1 t + \psi_1)} \rightarrow \psi'(t) = \frac{\boldsymbol{cos}^2\psi(t)}{\boldsymbol{sin}^2(\omega_1 t + \psi_1)}\omega_1 = \omega_1$$

Method 3 using direct derivative of phase function:

$$\omega(t) = \frac{d\psi(t)}{dt} = \frac{d}{dt}\left(\omega_1 t + \psi_1 - \frac{\pi}{2}\right) = \omega_1$$

Comments: Obtained derivation underline constant character of angular frequency of investigated monocomponent signal. However, taking into consideration mentioned practical representation of instantaneous phase function as periodical form limited in range $\langle -\pi, \pi \rangle$ with “artificial” phase jump from π to $-\pi$ in time instant equals t_1 ,we have to consider not only constant function ω_1 but also delta Dirac function in t_1.

TABLE II

ANALYTIC DERIVATION OF INSTANTANEOUS AMPLITUDE AND FREQUENCY OF: $x(t) = A_1 \sin(\omega_1 t + \psi_1) + A_2 \sin(\omega_2 t + \psi_2)$

Referring to above steps for monoconponent the instantaneous amplitude, phase and frequency of dual sine signal can be presented as:

$$z(t) = x(t) + jy(t) = A_1 \boldsymbol{sin}(\omega_1 t + \psi_1) + A_2 \boldsymbol{sin}(\omega_2 t + \psi_2) - j\left[A_1 \boldsymbol{cos}(\omega_1 t + \psi_1) + A_2 \boldsymbol{cos}(\omega_2 t + \psi_2)\right]$$

$$\rightarrow IA\boldsymbol{(}t\boldsymbol{)} = |z(t)| = \sqrt{A_1^2 + A_2^2 + 2A_1A_2 \boldsymbol{cos}\left((\omega_2 - \omega_1)t + (\psi_2 - \psi_2)\right)}$$

$$\rightarrow IP(t) = arctg\left(\frac{y(t)}{x(t)}\right) = arctg\left(-\frac{A_1 \boldsymbol{cos}(\omega_1 t + \psi_1) + A_2 \boldsymbol{cos}(\omega_2 t + \psi_2)}{A_1 \boldsymbol{sin}(\omega_1 t + \psi_1) + A_2 \boldsymbol{sin}(\omega_2 t + \psi_2)}\right)$$

$$\rightarrow IF(t) = \frac{1}{2\pi}\frac{A_1^2\omega_1 + A_2^2\omega_2 + (A_1A_2)(\omega_2 + \omega_1)\left(\boldsymbol{cos}\left((\omega_2 - \omega_1)t + (\psi_2 - \psi_2)\right)\right)}{A_1^2 + A_2^2 + 2A_1A_2 \boldsymbol{cos}\left((\omega_2 - \omega_1)t + (\psi_2 - \psi_2)\right)}$$

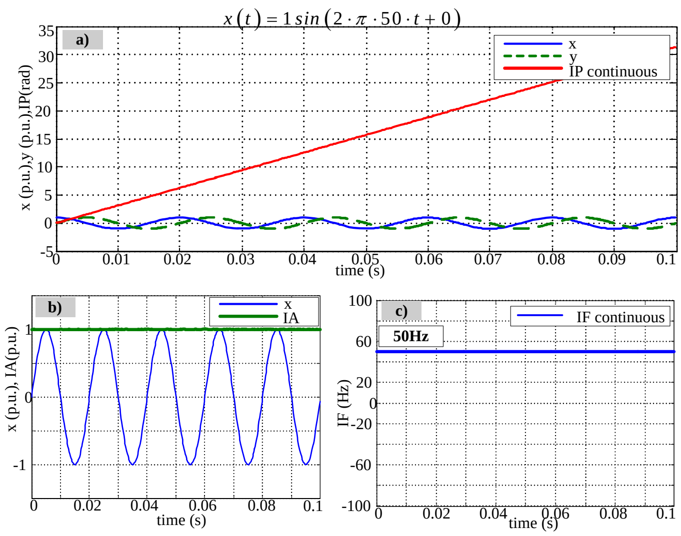

Fig. 3. Example of continuous calculation of Hilbert transform of monocomponent signal: (a) investigated signal $x(t)$ and its orthogonal form $y(t)$ with instantaneous phase IP(t);(b) signal and instantaneous amplitude IA(t);(c) instantaneous frequency IF(t)

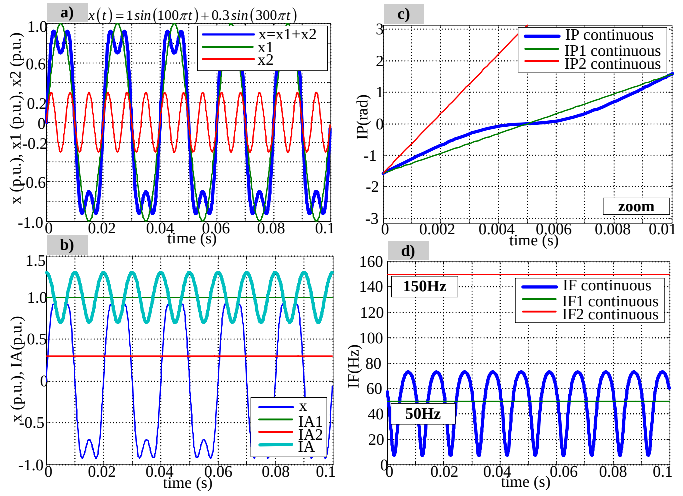

Fig. 4. Example of continuous calculation of Hilbert transform of dual component signal: (a) investigated signal $x(t)$ and its components $x_1(t)$ and $x_2(t)$; (b) instantaneous amplitude IA(t);(c) instantaneous phase IP(t);(d) instantaneous frequency IF(t)

True instantaneous amplitude, phase and frequency normally is not weighted by numerical limitation and can be rather interpreted as continuous functions. In comparison to Fig 1 we have constructed Fig. 3, presenting Hilbert transform and further calculation of instantaneous characteristics of signal parameters after correction of the phase function in continues mode, without the limitation $\langle -\pi,\pi \rangle$. Presenting trend represents intuitive expectation for constant amplitude and frequency of investigated monocomponent sine signal.

B. *Dualcomponent sinusoidal case*

In order to further exploration of Hilbert idea we have also reintroduce previous derivation into sum of sinus signals $x(t)=A_1\,sin(\omega_1 t+\psi_1)+A_2\,sin(\omega_2 t+\psi_2)$. Obtained formulas describing instantaneous amplitude, phase, and frequency are grouped in Table II. Mentioned instantaneous characteristic of signal parameters is also shown in Fig. 4, including comparison with instantaneous characteristics of particular signal components. We can distinguish general comment that direct application of Hilbert transform, and its further utilization to instantaneous parameters calculation, is not sufficient for distorted signal. The initial decomposition process is required preceded the Hilbert calculation. Having particular components of the signal we can perform Hilbert tool and obtain complex information about its time-varying aspects.

III. Conclusion

Hilbert transform is a one of the tool for creation of analytic signal which can be treated as initial form of the signal for calculation of instantaneous characteristic of the signal as instantaneous amplitude, phase or frequency. Presented in the paper numerical application confirmes some limitation and restriction of direct application of the Hilbert method. The solution is supporting the Hilbert calculation by previous decomposition of the signal. Application of Hilbert tool, and calculation of instantaneous parameters for every decomposed components can bring complex information about time-varying nature of investigated distortion.

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